Journal of Statistical Physics, Vol. 32, No. 1, 1983

Abstracts of the Workshop on Statistical Mechanics, Dynamical Systems, and Turbulence[1]

September 7–17, 1982

Oscar E. Lanford, Coordinator

INTRODUCTION

The Fall 1982 program of the Institute for Mathematics and its Applications was devoted to Statistical Mechanics and Dynamical Systems and was coordinated by Oscar Lanford. The program began with a workshop which consisted of a number of lecture series to introduce the participants to the areas of statistical mechanics, dynamical systems and turbulence.

Schematically, the organization of the lectures was as follows: In statistical mechanics, the lectures by Lanford and Gross were an elementary introduction to the general principles of equilibrium thermodynamics and statistical mechanics including Gibbs ensembles, partition functions, thermodynamic limits, and Gibbs states for infinite systems. The general theory was further developed in the lectures of Newman on correlation inequalities and the Lee–Yang theorem and those of Faris on proofs of existence of phase transitions. Tracy described what is known about exactly soluble models, especially the two-dimensional Ising model, and Halley surveyed critical phenomena, scaling, and renormalization group theory from the physicist's point of view.

In dynamical systems: Eckmann gave a general introduction to how dynamical systems ideas are applied in physics, and McGehee discussed concretely how ideas about exponential growth and decay of separation of nearby orbits (hyperbolicity) can be used to analyze the complicated motion produced by a particular equation. Joseph's lecture developed the

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0022-4715/83/0700-0169$03.00/0

relation between the mathematics of the Navier–Stokes equation and the physics of turbulence, and Lundgren surveyed the phenomenology of turbulence from a physical and engineering point of view. The lectures by Aronson and Collet were elementary mathematical introductions to the theory of bifurcations and of iteration of one-dimensional maps respectively. Tresser surveyed the phenomena that have been discovered by numerical experiments on the simple models of Hénon and Lorenz. On a more advanced level, Gallavotti discussed the Kolmogorov–Arnold–Moser theorem in the context of a general perturbation theory for Hamiltonian systems, and Conley presented a novel approach to the theory of hyperbolic sets and structural stability.

We present here abstracts of these lecture series together with reading lists in the hope that they will provide a useful guide to others who wish to learn these subjects.

The Institute for Mathematics and its Applications is sponsored by the National Science Foundation. Additional support for Institute activities has come from the Air Force Office of Scientific Research, the Army Office of Research, Cray Research, Honeywell, Magnetic Controls and 3M Corporation, as well as the following Participating Institutions of the Institute: Indiana University, Michigan State University, Northern Illinois University, Northwestern University, Ohio State University, Purdue University, University of Chicago, University of Illinois (Urbana), University of Iowa, University of Michigan, University of Minnesota.

INSTABILITY AND TRANSITION TO TURBULENCE

D. D. Joseph

University of Minnesota

The Navier–Stokes equations in a bounded domain have the special property that steady-state solutions are uniquely determined by the prescribed boundary conditions and prescribed forces when the Reynolds number is small. For larger values of the Reynolds number the solution, which is unique and stable when the Reynolds number is small, loses stability to a new motion with less symmetry. For instance, the new solution may break spatial symmetry, or time-dependent solutions may bifurcate from steady ones. In some problems a stable branch of the new solution bifurcates above the critical Reynolds number. Then the new motion can be a small amplitude perturbation of the old one. In the second case the new solution bifurcates below the critical Reynolds number and is usually unstable when the amplitude is small. This second (subcritical) case is sometimes associated with the direct transition to turbulence.

In the supercritical case it is possible to get repeated branching into higher-dimensional attractors. Landau and Hopf thought that these higher-dimensional attractors were tori for quasiperiodic attractors. They thought that this type of behavior might describe turbulence, but such solutions do not have a decaying autocorrelation and cannot describe turbulence.

The ideas of Ruelle, Takens, and Lorenz about the transition to turbulence are in better agreement with observations. After a few bifurcations we can have attracting sets which are not multiperiodic and which have continuous spectra, decaying autocorrelations, and other features of observed turbulence. The best agreement between theories of the Ruelle–Takens type and experiments is for small systems with widely separated eigenvalues whose dynamics are actually governed by a small number of modes.

There is good evidence that the dynamics of the Navier–Stokes equations are governed by a finite number of ODEs (Foias and Prodi, Foias and Teman, see Refs. 2 and 3) which arise from Galerkin methods after truncating. The problem is that the truncation number is evidently an increasing function of the Reynolds number and may be very large.

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3. G. I. Barenblatt, G. Iooss, and D. Joseph (eds.), *Nonlinear Dynamics and Turbulence* (Pitman, London, 1983).

Other references can be found in these three.

THE PHYSICS AND THEORY OF TURBULENCE

Thomas Lundgren

University of Minnesota

The lecture was organized into four parts. The first part consisted of slides of a number of flow visualizations of fully developed turbulent flows. The occurrence of both spatial and temporal chaos at small scales was emphasized.

The second part was a discussion of the cascade of energy from large scales to small scales and the Kolmogorov theory of the energy spectrum.

Thirdly, the use of an eddy viscosity, the classical practical solution to the problem of chaos at small scales, was described. The state of the art of eddy viscosity models for numerical simulation of turbulent flows was surveyed.

Finally, a model of the small-scale structure of high Reynolds number turbulent flows was briefly described.

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4. T. S. Lundgren, Strained Spiral Vortex Model for Turbulent Fine Structure, *Phys. Fluids*, **12**:2193 (1982).

DYNAMICAL SYSTEMS APPROACH TO TURBULENCE

J.-P. Eckmann

University of Geneva

In this introductory lecture, the main motivations of dynamical systems theory are outlined and illustrated. One studies general properties of differential equations of the form $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t))$, where $\mathbf{x}, \mathbf{F} \in \mathbb{R}^n$, or of maps $\mathbf{x}_{n+1} = \mathbf{G}(\mathbf{x}_n)$, $n \in \mathbb{Z}^+$. When such equations are dissipative, i.e., $\operatorname{div} \mathbf{F} < 0$, $|\det d\mathbf{G}| < 1$, the motion of a typical initial point approaches an attractor. In the absence of a classification of attractors, scenarios are outlined for one-parameter families of maps and flows, leading from trivial attractors to more complicated ones. In particular the period-doubling scenario is emphasized.

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J.-P. Eckmann, *Rev. Mod. Phys.* **53**:643–654 (1981).

ONE-DIMENSIONAL MAPPINGS

Pierre Collet

Ecole Polytechnique

This talk reviewed some of the recent results about iterations of unimodal maps of an interval. The first part was devoted to topological questions. The theory of kneading sequences, a topological invariant, was described. Much information about the dynamics can be obtained from this invariant if the Schwarzian derivative $(f'''/f') - 3/2(f''/f')^2$ is negative. For example, one can decide if there is a stable periodic orbit, if two maps with no stable periodic orbits are topologically conjugated, or if a given map is topologically equivalent to a piecewise linear one. Using the notions of * operation, and maximal sequences, one can derive some properties of the doubling operation $f \to f \circ f$.

The second part of this talk was devoted to the study of universal properties. Some numerical and experimental results were briefly described. The technique of renormalization group analysis was discussed. In this case, the renormalization transformation is composition followed by a scaling. One can explain universality for infinite sequences of period doubling bifurcations, the inverse cascade, and many other similar phenomena.

In the last part of this talk, some results about invariant measures and ergodic theory were described. The definitions of sensitive dependence on initial conditions, and Lyapunov exponents were discussed and illustrated by some numerical examples. The abundance of stochastic behavior in one-parameter families, and some criteria for the existence of an invariant measure absolutely continuous with respect to the Lebesgue measure were briefly discussed. Finally, the results for unimodal maps were compared to those for everywhere expanding maps.

A large part of the results described in these talks can be found in Ref. 1.

REFERENCE

1. J.-P. Eckmann and P. Collet, *Iterated Maps on the Interval as Dynamical Systems* (Birkhauser, Boston, 1980).

INTRODUCTION TO BIFURCATION THEORY FOR MAPS

D. G. Aronson

University of Minnesota

Let $T_\mu(x): \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be a smooth function of (x, μ) and suppose that T_0 has a fixed point $x = x_0$, i.e., that

$$T_0(x_0) = x_0$$

We shall assume throughout that T_μ is a diffeomorphism. It is well known that x_0 is stable if

$$\sigma(D_x T_0(x_0)) \subset U \equiv \{z \in \mathbb{C} : |z| < 1\},$$

where $\sigma(A)$ denotes the spectrum of A. Roughly speaking, x_0 is stable means that $T_0^n(x) \equiv (T_0 \circ T_0 \circ \cdots \circ T_0)(x) \to x_0$ as $n \to \infty$ for all x in some neighborhood of x_0. By the Implicit Function Theorem, there is a function $\mu \to x(\mu)$ defined for μ near 0 such that $x(0) = x_0$ and $T_\mu(x(\mu)) = x(\mu)$ provided that $D_x T_0(x_0) - I$ is invertible. The invertibility condition is satisfied if $\sigma(D_x T_0(x_0)) \subset U$. Now suppose there exists a $\mu^* > 0$ such that $x(\mu)$ is defined on $[0, \mu^*)$, $\lim_{\mu \uparrow \mu^*} x(\mu) \equiv x^*$ exists and is in the domain of T_{μ^*}, and $\sigma(D_x T_{\mu^*}(x^*)) \cap \partial U \neq \emptyset$. Thus either $x = x(\mu)$ cannot be extended beyond $\mu = \mu^*$ or else it can be extended but loses its stability. Bifurcation theory for maps provides a description of the behavior of the invariant set for T_μ for μ near μ^*.

Case 1. Saddle-Node Bifurcation. Suppose that

$$\sigma(D_x T_\mu(x(\mu))) = \{\lambda_\mu\} \cup R_\mu$$

where $R_\mu \subset U$ for $\mu \in [0, \mu^*]$, $|\lambda_\mu| < 1$ for $\mu \in [0, \mu^*)$, and $\lambda_{\mu^*} = 1$, i.e., at $\mu = \mu^*$ a single eigenvalue of $D_x T_\mu(x(\mu))$ hits the unit circle at 1 and $x = x(\mu)$ cannot be extended beyond $\mu = \mu^*$. Using the Center Manifold Theorem(3) we can reduce to a one-dimensional problem. Specifically, let $\mathbf{e}$ be a nonzero vector such that

$$D_x T_{\mu^*}(x^*)\mathbf{e} = \mathbf{e}$$

Then there exists a neighborhood V of (x^*, μ^*) in $\mathbb{R}^d \times \mathbb{R}$ and a two-dimensional manifold M tangent to the plane of $\mathbf{e}$ and the μ-axis at (x^*, μ^*) which is locally invariant and locally attractive. For each μ near μ^* let M_μ denote the μ-section of $M \cap V$ and consider the map $G_\mu \equiv T_\mu \mid M_\mu$: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Observe that

$$G_{\mu^*}(y^*) = y^* \quad \text{and} \quad D_y G_{\mu^*}(y^*) = 1$$

where $y^* = x^* \mid M_{\mu^*}$. Although we cannot solve $G_\mu(y) = y$ for y as a function of μ in a neighborhood of μ^*, we can solve for μ as a function of y provided that

$$D_\mu G_{\mu^*}(y^*) \neq 0$$

Indeed the result is

$$\mu(y) = \mu^* + \tfrac{1}{2}(y - y^*)^2 \mu''(y^*) + \cdots$$

provided that

$$D_y^2 G_{\mu^*}(y^*) \neq 0 \qquad (*)$$

Here

$$\mu''(y^*) = D_y^2 G_{\mu^*}(y^*) / D_\mu G_{\mu^*}(y^*)$$

Our assumptions on $x = x(\mu)$ assure us that $\mu''(y^*) < 0$. Thus we get the following picture.

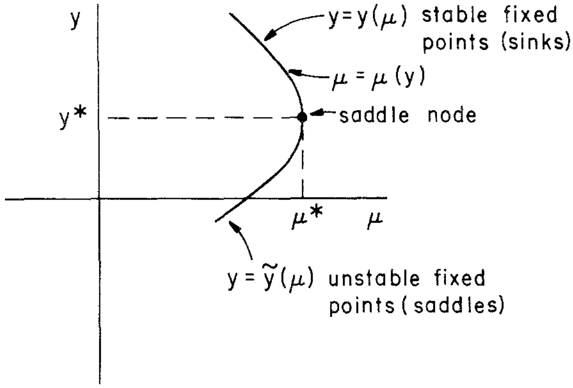

Fig. 1.

Case 2. Flip Bifurcation. In the notation of Case 1, suppose that $R_\mu \subset U$ for $\mu > 0$, $|\lambda_\mu| < 1$ for $0 < \mu < \mu^*$, and $\lambda_{\mu^*} = -1$. In this case $x = x(\mu)$ extends beyond $\mu = \mu^*$ but is not stable for $\mu > \mu^*$. We shall show that, under appropriate conditions, new invariant objects are created at $\mu = \mu^*$ which take up the stability lost by $x(\mu)$. Again we use the Center Manifold Theorem to reduce to a one-dimensional map $y \to G_\mu(y)$ with

$$G_{\mu^*}(y^*) = y^* \quad \text{and} \quad D_y G_{\mu^*}(y^*) = -1$$

By a translation of coordinates we can assume that the fixed point $y(\mu) \equiv 0$. Assume that

$$D_\mu D_y G_{\mu^*}(0) \neq 0$$

For example, if it is negative then $D_y G_\mu(0)$ decreases as μ increases through μ^*. Reparametrize so that

$$D_y G_\mu(0) = -(1 + \mu - \mu^*)^{1/2}$$

for $|\mu - \mu^*|$ sufficiently small and consider the second iterate

$$H_\mu(y) \equiv (G_\mu \circ G_\mu)(y)$$

The idea is to show that H_μ has nonzero fixed points for $\mu > \mu^*$ and that these points are not fixed points of G_μ.

Some computation shows that $D_y H_{\mu^*}(0) = 1$ but $D_y^2 H_{\mu^*}(0) = 0$. Thus condition (*) is violated and we do not have a generic saddle-node bifurcation. Further computations using Taylor's Theorem and the Implicit Function Theorem show that for sufficiently small $\mu > \mu^*$, H_μ has fixed points

$$y_\mu^\pm = \pm\left(\frac{\mu - \mu^*}{|a|}\right)^{1/2} + O(\mu - \mu^*)$$

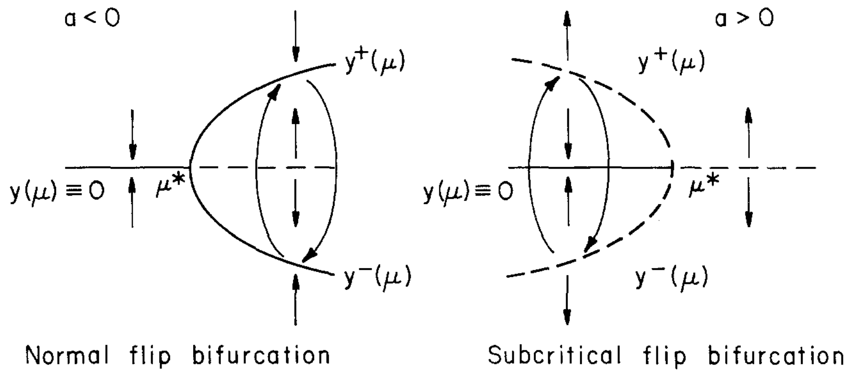

Fig. 2.

provided that

$$a = \frac{1}{3!}\left\{-2D_y^3G_{\mu^*}(0) - 3\left[D_y^2G_{\mu^*}(0)\right]^2\right\} \neq 0$$

The situation is shown schematically in Fig. 2. Moreover these points are stable (unstable) if $a < 0$ (> 0) and satisfy

$$G_\mu(y_\mu^\pm) = y_\mu^\mp$$

i.e., the $y_\mu^\pm$ are period 2 points for G_μ.

Case 3. Hopf Bifurcation. Suppose that

$$\sigma(D_x T_\mu(x(\mu))) = \{\lambda_\mu, \bar{\lambda}_\mu\} \cup R_\mu$$

with $R_\mu \subset U$ and $\lambda_\mu \neq \bar{\lambda}_\mu$ for all μ near μ^*. Suppose further that $|\lambda_\mu| < 1$ for $\mu < \mu^*$, $|\lambda_{\mu^*}| = 1$, and $|\lambda_\mu| > 1$ for $\mu > \mu^*$. Again $x = x(\mu)$ can be continued through $\mu = \mu^*$ but the fixed point loses stability. The Center Manifold Theorem permits us to reduce to a problem for a two-dimensional map. Specifically, the center manifold M is tangent to the eigenspace of $\{\lambda_{\mu^*}, \bar{\lambda}_{\mu^*}\}$ and the μ-axis at (x^*, μ^*) and we work with the map $G_\mu \equiv T_\mu | M_\mu : \mathbb{R}^2 \to \mathbb{R}^2$. Translate the coordinates so that the fixed points $y(\mu) \equiv x(\mu) | M_\mu \equiv 0$.

(I) If $(d/d\mu)|\lambda_{\mu^*}| > 0$ and $\lambda_{\mu^*}^k \neq 1$ for $k = 1, 2, 3$, or 4 there exists a μ-dependent change of variables on $\mathbb{R}^2$ such that

$$G_\mu : (r, \phi) \to \big((1 + \mu - \mu^*)r + f_1(\mu - \mu^*)r^3 + \cdots,$$

$$\phi + \theta(\mu - \mu^*) + f_2(\mu - \mu^*)r^2 + \cdots\big)$$

where (r, ϕ) are polar coordinates on $\mathbb{R}^2$.

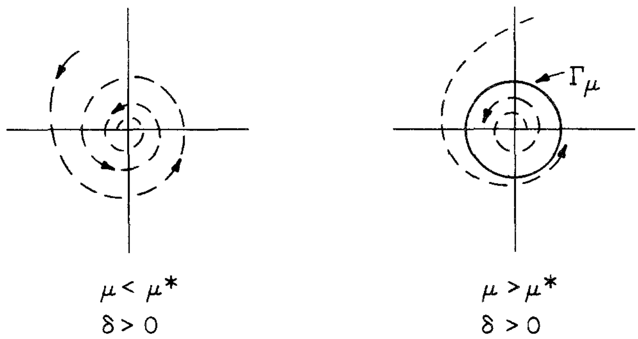

Fig. 3.

(II) Set $\delta \equiv f_1(0)$. If $\delta > 0$ then, for sufficiently small $\mu > \mu^*$, G_μ has a one-dimensional attracting set Γ_μ which is topologically a circle. If $\delta < 0$ the invariant set Γ_μ is not attracting. In either case, radius $\Gamma_\mu \sim (\mu - \mu^*)^{1/2}$.

Note that Γ_μ is not an orbit: it is an invariant set. For some values of μ there may be periodic orbits in Γ_μ.

Applications. Consider the system of ordinary differential equations

$$\dot{x} = F_\mu(x) \qquad (**)$$

with $x, F_\mu \in R^d$ and $F_\mu(x)$ smooth. Suppose that for $\mu = 0$ there is a periodic orbit $x = \phi_0(t)$ with period T, i.e., $\phi_0(t + T) = \phi_0(t)$ for all $t \in R$. Fix a point ξ on the orbit and take a small $(d-1)$-dimensional surface Σ which is transverse to the orbit at ξ. Define the map $P_\mu : \Sigma \times \mathbb{R} \to \Sigma$ as follows: For each $\zeta \in \Sigma$ and sufficiently small $\mu \in \mathbb{R}$ let $P_\mu \zeta = \phi_\mu(\hat{t}; \zeta)$ where $\phi_\mu(\cdot\,; \zeta)$ denotes the solution of (**) through the point $(\zeta, 0)$ and

$$\hat{t} = \inf\{t > 0 : \phi_\mu(t; \zeta) \in \Sigma\}$$

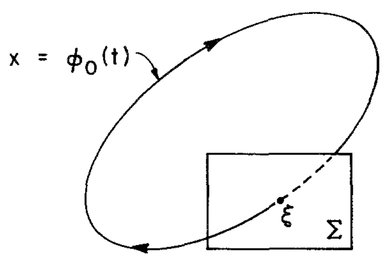

Fig. 4.

By standard results in the theory of ordinary differential equations $\hat{t} < +\infty$ and P_μ is well defined for all $(\zeta, \mu) \in V \times (-\delta, \delta)$, where V is a neighborhood of ξ in Σ. In view of the uniqueness of the solution of (**), P_μ is a diffeomorphism. P_μ is called the Poincaré map.

Observe that ξ is a fixed point of P_0. If

$$1 \notin \sigma(D_\zeta P_0 \xi)$$

then the Implicit Function Theorem implies the existence of a $\zeta = \zeta(\mu)$ such that $\zeta(0) = \xi$ and $P_\mu \zeta(\mu) = \zeta(\mu)$. By the definition of P_μ there exists a $\hat{t} \in \mathbb{R}^+$ such that $\phi_\mu(\hat{t}; \zeta(\mu)) = \zeta(\mu) = \phi_\mu(0; \zeta(\mu))$. Therefore the periodic orbit persists for μ near 0.

If

$$\sigma(D_\zeta P_0 \xi) \subset U$$

then the orbit $x = \phi_0(t)$ is attracting. As μ increases from 0 either $\zeta = \zeta(\mu)$ leaves the domain of P_μ or else the spectrum hits ∂U. We consider briefly the latter case. In what follows we assume tacitly that all the appropriate generic conditions are satisfied.

(1) $\sigma(D_\zeta P_\mu \zeta(\mu)) = \{\lambda_\mu\} \cup R_\mu$ *with* $R_\mu \subset U$. (a) Saddle-Node Bifurcation: $|\lambda_\mu| < 1$ for $\mu < \mu^*$, $\lambda_{\mu^*} = 1$. For $\mu < \mu^*$ there are a stable and an unstable orbit, as shown in Fig. 5. The orbits coincide for $\mu = \mu^*$ and there are no periodic orbits for $\mu > \mu^*$.

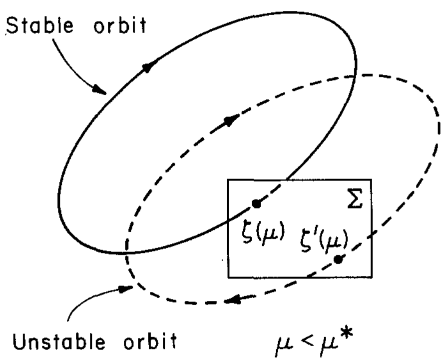

Fig. 5.

(b) Flip Bifurcation: $|\lambda_\mu| < 1$ for $\mu < \mu^*$, $\lambda_{\mu^*} = -1$. Without loss of generality we can assume that $\zeta(\mu) \equiv \xi$.

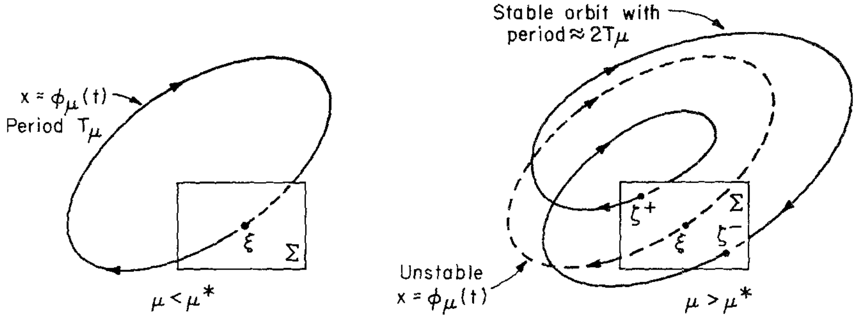

Fig. 6.

(2) Hopf Bifurcation. $\sigma(D_\zeta P_\mu \zeta(\mu)) = \{\lambda_\mu, \bar{\lambda}_\mu\} \cup R_\mu$ with $R_\mu \subset U$. Suppose $|\lambda_\mu| < 1$ for $\mu < \mu^*$, $|\lambda_{\mu^*}| = 1$, $|\lambda_\mu| > 1$ for $\mu > \mu^*$, and $\lambda_{\mu^*}^k \neq 1$ for $k = 1, 2, 3, 4$. Again change variables so that $\zeta(\mu) \equiv \xi$.

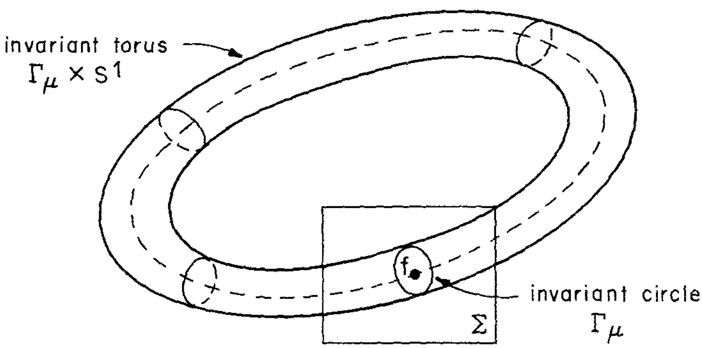

Fig. 7.

Notes on Sources. The idea of reducing bifurcation problems in $\mathbb{R}^d$ (or infinite-dimensional spaces) to the lower-dimensional center manifold seems to be due to Ruelle and Takens.(6) A very elegant account of their methods can be found in Lanford's paper,(4) the book of Marsden and McCracken,(5) and (in somewhat less detail) Henry's monograph.(2) Reference 5 also contains a detailed account of the Hopf Bifurcation Theorem for flows which we have not touched on here. Arnol'd's paper(1) gives an account of the Hopf Bifurcation Theorem for two-parameter families of diffeomorphisms. Alternative approaches to bifurcation theory which use the Liapunov–Schmidt method rather than the Center Manifold Theorem can be found in the books of Sattinger[7] and of Iooss and Joseph,[3] as well as in the paper of Weinberger.[8]

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QUALITATIVE THEORY OF HAMILTONIAN SYSTEMS

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University of Rome

I introduce the notion of relative integrability for pairs of Hamiltonian systems.

The first illustration of this notion is the analysis of the problem of how to conjugate a small perturbation of a system which is integrable by quadratures with another system which is also integrable by quadratures. I discuss through a few examples why this problem has a complex answer and why such an answer reflects only partially some naive intuitions. The examples are adapted from the classical works of Poincare and Birkhoff (References: A).

After stating various forms of the KAM theorem I try to make it clear that it is a theorem of "dimensional nature" illustrating in detail the essential points of its proof and deriving, at the same time, some smoothness properties of the family of invariant tori whose existence is proven by the theorem. Then I discuss a few aspects of the problem of finding conditions guaranteeing the integrability by quadratures of a perturbation of a nonresonant harmonic oscillator and I present a sufficient condition for the convergence of the Birkhoff series (References: B).

Finally I discuss the theory of perturbations for systems which are not integrable by quadratures considering the case of the perturbations of the geodesic flow on a surface of constant negative curvature. I present some recent results. The first example is on the sufficiency of the existence of formal series solutions of the Hamilton–Jacobi equation (for the conjugation of the perturbed system with the unperturbed one) for the actual solubility of the equation. The second example is a description of a complete set of invariants for the conjugation of perturbed and unperturbed geodesic flows (References: C).

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REMARKS ON CHAIN RECURRENCE AND HYPERBOLICITY

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The representation of a flow on a compact space as the extension of a chain-recurrent flow by a strongly gradientlike flow was discussed first. Pursuing the gradientlike flow one finds things like the Morse theory of critical points of gradient flows or more generally, of isolated invariant sets. Following the chain-recurrent part, one is led, among several other things, to hyperbolic invariant sets. The basic dynamical results for hyperbolic sets were then discussed in terms of Wazewski's principle. The ideas of proofs of the structural stability, the shadowing lemma, and the stable manifold theorem were based on the construction of tubes of uniformly small diameter about orbits of the displacement equations. These tubes are "Wazewski sets" and their "nontriviality" together with the near linearity of the displacement equations gives the results.

Finally, a way of abstracting the ideas of hyperbolic sets to arbitrary invariant sets of flows on compact metric spaces which have a global surface of section was outlined. The result is that tubes can always be constructed about bundles of orbits which are "held close together" by the flow. In this approach, the ideas of isolation and index, seen in the general Morse theory, appear again.

A HYPERBOLIC INVARIANT SET FOR A FORCED PENDULUM

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The following differential equation describes a periodically forced pendulum:

$$\ddot{\theta} = -\alpha\mu\dot{\theta} - (1 + 2\mu\cos 2t)\sin\theta \qquad (*)$$

Note that, for $\mu = 0$, this equation models an unforced rigid pendulum with angular displacement θ measured from the bottom of its swing. The parameter μ measures the amplitude of the periodic forcing term, while the parameter α is related to the damping. A theoretical physicist who regrets that he was taken seriously has proposed equation (*) as a model for an acrobatic act.

Consider a solution $\theta(t)$ which has only simple zeros, i.e., the pendulum has a nonzero angular velocity whenever it is at the bottom of its swing. Since these zeros are isolated, they can be numbered

$$\cdots < t_{-1} < t_0 < t_1 < t_2 \cdots$$

Define

$$\sigma_n(\theta) \equiv \begin{cases} +1 & \text{if} \quad \dot{\theta}(t_n) > 0 \\ -1 & \text{if} \quad \dot{\theta}(t_n) < 0 \end{cases}$$

Note that σ_n is positive if the pendulum is moving counterclockwise at the nth time it hits the bottom of its swing, while σ_n is negative if the pendulum is moving clockwise at that time. The sequence $\sigma_n(\theta)$ can be called the *itinerary* of the solution θ. One can prove the following theorem, which states that every itinerary is achieved.

Theorem. Fix α so that $|\alpha| < 1/2\pi \sinh \pi$ (≈ 0.01378). For sufficiently small fixed $\mu \neq 0$, the following statement holds. For each bi-infinite sequence $\tau_n = \pm 1$, there is a solution θ of (1) such that

$$\sigma_n(\theta) = \tau_n \qquad \text{for all } n$$

The key to the proof is to construct an appropriate hyperbolic invariant set. One writes equation (*) as a first-order system by introducing the variable $\omega \equiv \dot{\theta}$.

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\sin\theta - \mu(2\cos 2t \sin\theta + \alpha\omega)$$

This system has a periodic orbit of period π at $(\theta, \omega) = (\pi, 0)$. For small μ, this orbit is hyperbolic with a two-dimensional stable manifold and a two-dimensional unstable manifold. For $\mu = 0$, these two manifolds coincide exactly. For small $\mu \neq 0$, the method of Mel'nikov[2] can be used to establish the existence of two nondegenerate homoclinic orbits, one passing near $(\theta, \omega) = (0, 2)$, and the other passing near $(\theta, \omega) = (0, -2)$. The union of the two homoclinic orbits with the periodic orbit is a hyperbolic invariant set and hence has the shadowing property.[1] The proof of the theorem thus is reduced to constructing pseudo-orbits corresponding to the specified sequence τ_n. For $\tau_n = +1$, we choose the pseudo-orbit to follow the homoclinic orbit near $(0, 2)$, while, for $\tau_n = -1$, we choose it to follow the homoclinic orbit near $(0, -2)$. The shadow of this pseudo-orbit is the orbit with the desired itinerary.

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THE EXAMPLES OF LORENZ AND HÉNON

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The Lorenz equations read:

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy\end{aligned} \tag{1}$$

where σ, b, r are parameters generally chosen in R^{+*}. Although obtained as a result of cutting off Galerkin-type equations for thermal convection between planes, they do not describe the turbulence of a convective flow but constitute an interesting model "per se." One generally varies one parameter in (1) and, varying r for $\sigma = 10$, $b = 8/3$ (the most studied case), one has the following succession of events:

$r < 1$: O is the unique critical point. It is stable and attracts all orbits.

$r = 1$: Pitchfork bifurcation at O: two critical points $O^{\pm} = (\pm[b(r-1)]^{1/2}, \pm[b(r-1)]^{1/2}, r-1)$ branch off: this nongeneric bifurcation is due to the invariance of (1) under the symmetry $(x, y, z) \to (-x, -y, z)$.

$1 < r < r_h \cong 13.926$: The two branches $W_{\text{loc}}^{\pm u}(O)$ of the unstable manifold of O are attracted by the closest points $O^{\pm}$. (The stable manifold of O is like a barrier between the basins of $O^{\pm}$.)

$r = r_h + \epsilon$ (ϵ small enough): There has been an abrupt transition: The topological entropy of a typical first return map was 0 for $r < r_h$ and is $\log 2$ for $r = r_h + \epsilon$: Furthermore $W_{\text{loc}}^{\pm u}(O)$ is now attracted by $O^{\mp}$. It seems that, except for a set of Lebesgue measure 0, all points still have orbits converging to O^- or O^+

$r_A \cong 26.06$: The support of positive topological entropy gets positive Lebesgue measure.

$r_A < r < r_D$: The topological entropy is smaller than $\log 2$ on a typical first return map but one seemingly has a strange attractor, which is in concurrence with the sinks $O^{\pm}$ up to $r = r_H \cong 24.74$.

$r = r_H \cong 24.74$: $O^{\pm}$ undergo subcritical Hopf bifurcations. The symmetric one-parameter families of unstable cycles thereby generated can be numerically traced back to the pair of homoclinic orbits one observes for $r = r_h$. [Note that for other values of (σ, b), $r_H(\sigma, b)$ corresponds to supercritical Hopf bifurcations: then for some $r_{s \cdot n} > r_H$, the stable cycles thereby generated disappear in saddle node bifurcations when they encounter the unstable cycles generated by the bifurcation at r_h.]

$r_H < r < r_D$: One seemingly has an indecomposable strange attractor, which attracts all orbits except for the stable manifold of the critical points, a set of Lebesgue measure zero.

$r > r_D$: A very complicated succession of events occurs, including cascades of period doublings, intermittency, observation of stable cycles, and horseshoe-type first return maps.

It seems that one can understand the dynamics for $1 < r < r_D$ by the fact that (1) preserves some strong stable foliation: this would allow one to reduce the dynamics of some first return map to some one-dimensional map obtained by projection along the foliation: indeed modeling the one-dimensional maps allows one to construct other well-understood flows.

It seems that the complexity for $r > r_D$ can be understood in terms of the existence, for some r_{het}, of a heteroclinic connection between the saddle foci $O^{\pm}$. If such a heteroclinic connection does exist, typical first return maps present infinitely many horseshoes. The complex behavior observed for $r > r_{\text{het}}$ and $r_D < r < r_{\text{het}}$ corresponds to destructions and constructions of horseshoes.

Hénon proposed the mapping

$$(x, y) \to (1 - ax^2 + y, bx) \tag{2}$$

as a simple model for the formation of horseshoes. For many sets (a, b), one observes numerically what seem to be strange attractors. These cannot be hyperbolic strange attractors, and one has no serious indications that what ones sees is not merely due to the impossibility for a computer to resolve the subtleties of such a map; e.g., tiny stable cycles of low period ($\sim$30) could be invisible with 16-digit computations. Although one knows that period-doubling cascades occur for (2) when varying a (b fixed), one does not know if the variation from zero to positive topological entropy is due to the cascade, as is proven for $b = 0$ where (2) reduces to a one-parameter family of one-dimensional endomorphisms. One only knows that for some (a, b), (2) has positive topological entropy and that, at least for b small enough, for some (a, b), (2) possesses infinitely many concurrent sinks. Recent investigations of other models for the formation of a horseshoe seem, however, to indicate that the occurrence of infinitely many sinks does not account for the structure of the seemingly strange attractors observed numerically. A clear understanding of the Hénon mapping would certainly be an important advance in dynamical systems theory, especially in regards to possible applications to the understanding of the transition to turbulence.

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INTRODUCTION TO STATISTICAL MECHANICS

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The main topics discussed were as follows:

(1) A review of the thermodynamics of simple substances. The objective of this part of the lectures was to describe, in a mathematically precise but nonaxiomatic way, the basic physical principles of thermodynamics. The discussion emphasized the importance of convexity properties of the thermodynamic functions and the use of Legendre transforms. The way first-order phase transitions manifest themselves in the thermodynamic functions was described carefully.

(2) Basic principles of statistical mechanics. The principal classical models for microscopic matter were surveyed, including both continuous and lattice systems. (For lack of time, quantum statistical mechanics was not discussed.) The microcanonical, canonical, and grand-canonical ensembles were introduced, and the prescriptions for computing thermodynamic functions as thermodynamic limits of logarithms of partition functions described.

(3) Thermodynamic limits of partition functions. The main ideas in the proof of the existence of the thermodynamic limit of the microcanonical entropy were sketched, and it was shown how to obtain the thermodynamic limits of the canonical and grand-canonical partition functions as an easy corollary of this result.

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STATES OF INFINITE SYSTEMS

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The laws of equilibrium thermodynamics (conservation of energy, increase of entropy, etc.) are idealized expressions of the directly observed behavior of large scale matter (chunks of about 10^{20} atoms or more). Statistical mechanics on the other hand attempts on the basis of a microscopic model of large scale matter to explain the general form of the laws of thermodynamics, as well as to predict the value of specific thermodynamic quantities (such as specific heat) for a specific substance.

After a brief survey of the basic features of equilibrium thermodynamics, it was shown how these features can be recovered from statistical mechanics. The mechanism was carried out in the technically relatively simple context of a crystal. Because a thermodynamic system consists of a large number of particles, the customary mathematical idealization of this is an infinite system. For a microscopic model of an infinite crystal based on classical mechanics (rather than quantum mechanics) the state of the system is given by a measure on an infinite product space. The notions of entropy and free energy were defined and their relations explained in this context.

These lectures were based in large part on the books of Ruelle[3,4] and Israel.[2] For a survey adhering close to the spirit of these lectures see Ref. 1, which contains also further references.

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CORRELATION INEQUALITIES AND THE LEE–YANG THEOREM

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We consider the finite Ising models $\{s_i : i \in \Lambda \subset Z^d\}$ with probability distribution $d\nu(s) = Z^{-1}\exp[-\beta U(s)]\prod\rho_i(ds_i)$ where $-\beta U(s) = h\sum s_i + \sum J_{i-j}s_i s_j$ with $J_{i-j} > 0$.

The GKS inequalities, valid for even ρ_i's and $h \geqslant 0$, are stated and proved using Ginibre's duplicate variable methods. They can be used to prove existence of thermodynamic limits of correlations and to compare phase transitions of different models. They can also show that the magnetization, $m(h) = |\Lambda|^{-1}\int(\sum s_i)\,d\nu(s)$ is positive and increasing for $h \geqslant 0$.

The GHS inequality, valid for certain even ρ_i's and $h \geqslant 0$, is stated and proved by the Ellis–Monroe quadruplicate variable method; it implies that m is concave for $h \geqslant 0$ so that discontinuities of m can occur only at $h = 0$.

The FKG inequalities, valid for all ρ_i's, are stated and proved by a method using diffusion semigroups based on work of Pitt. They can be used to relate the asymptotic independence of the s_i's to the decay properties of $\langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle$ as $|i-j| \to \infty$ (in the infinite volume system).

The Lee–Yang theorem, valid for certain even ρ_i's, states that the zeros of Z in the complex h-plane are all purely imaginary; it implies that m is analytic for $\mathrm{Re}\,h \geqslant 0$ so that phase transitions can only occur (for real h) at $h = 0$. The Lieb–Sokal proof of the Lee–Yang theory is presented.

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FKG Inequalities

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Lee–Yang Theorem

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ORDER AT LOW TEMPERATURE

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INTRODUCTION

The purpose of these lectures is to explain how microscopic short-range forces can lead to macroscopic long-range order, even in the presence of randomness. The obvious mechanism is that the influence simply propagates from neighbor to neighbor through the macroscopic body. This is essentially correct, but there are a few subtle points to consider.

The first is that there is a qualitative change as the temperature (a measure of randomness) varies continuously. At high temperatures there is no macroscopic order. It is only below a certain critical temperature that the ordered phases appear.

The second point is that the situation depends strongly on the dimension d of space. When $d = 1$ the critical temperature is absolute zero and any amount of randomness destroys macroscopic order. The dimension of space must play a role in the analysis.

1. CONTOURS

The first method for attacking the problem is the use of *contours*. This dates back to a note of Peierls in 1936. The method gives a proof of the existence of multiple phases for the ferromagnetic Ising model.

This model is an idealization of a magnet. Let $\Lambda \subset Z^d$ be a finite set of integer lattice points; think of this as a crystal. Let $s: \Lambda \to \{\pm 1\}$ be a function; this is supposed to be a microscopic configuration of the magnet. Then $s(\mathbf{n})$ is interpreted as the value of the atomic spin at the site $\mathbf{n}$ in the crystal Λ.

The energy of a configuration s is defined to be

$$H(s) = \tfrac{1}{2} \sum_{|\mathbf{n}-\mathbf{m}|=1} |s(\mathbf{n}) - s(\mathbf{m})|^2$$

where the sum includes nearest-neighbor pairs $\{\mathbf{n}, \mathbf{m}\}$ in the crystal. Note that each such pair that is aligned contributes zero to the energy; each misaligned pair contributes 2. Thus alignment produces low energy.

The sum should also include nearest-neighbor pairs with only one member in the crystal Λ; we make the convention that $s(\mathbf{n}) = +1$ whenever

n is not in Λ. This is a boundary condition that tends to make positive spin configurations have somewhat lower energy.

The probability of a configuration is given by the Gibbs prescription as

$$P(\{s\}) = Z^{-1}\exp[-H(s)/T]$$

where T is the temperature parameter. The coefficient Z^{-1} is simply a normalization constant designed to make the probability of the set of all configurations come out to be one.

The main result says that if the dimension $d \geqslant 2$, then for every fixed **n** in Z^d,

$$\lim_{T\downarrow 0} \sup_{\Lambda} P(s(\mathbf{n}) = -1) = 0$$

Thus the plus boundary conditions propagate their influence into the interior of the crystal, no matter how large it may be and how far **n** may be from the boundary.

It follows in particular that for T sufficiently small

$$P(s(\mathbf{n}) = -1) \leqslant \epsilon < \tfrac{1}{2}$$

for all Λ. This behavior persists in the thermodynamic limit $\Lambda \to Z^d$. Thus the bulk magnet is magnetized in the plus direction. On the other hand, if we had used minus boundary conditions we would have gotten the opposite behavior. This shows that there are two phases in the thermodynamic limit.

It is important to stress that the result is false when the dimension $d = 1$. In this case an ordered state is a chain of aligned spins, and there are so many places the chain can break that there will surely be chains of spin up and chains of spin down in roughly equal proportion.

There is no point repeating the proof of the theorem in this outline. There are many accounts. One that I found particularly readable is in lectures by Spitzer.[(1)] The basic idea is that any site with spin that is not aligned with the spins on the boundary is surrounded by a contour separating adjacent misaligned spins. Long contours are improbable at low temperature. But when $d \geqslant 2$ there are relatively few short contours surrounding the site. Thus the nonconforming spin has small probability.

2. INFRARED BOUNDS

There is another method of proving the existence of multiple phases at low temperatures, the method of infrared bounds. In one respect this is less powerful, in that the results are only for dimension $d \geqslant 3$. On the other hand, the method gives a picture of phase transitions that applies not only to Ising models but also the continuous spin models. The picture that emerges is that $d=2$ is a borderline case, but multiple phases are to be expected as a matter of course in higher dimensions.

The original 1976 paper on infrared bounds is by Fröhlich, Simon, and Spencer.(2) This is still a very readable account. There is a more abstract and general treatment in a paper by Fröhlich, Israel, Lieb, and Simon,(3) and this should be consulted by anyone who wishes to apply the method. Their version uses generating functions (Laplace transforms) in a systematic way. The version I present uses only expectations of quadratic expressions and so is crude by contrast, but perhaps more elementary.

The framework is now an infinite crystal Z^d. A configuration s is a function from Z^d to the reals. There is a probability measure on the space of configurations, and E is used to denote expectation. The measure is assumed to be translation invariant.

The strategy is to choose one direction as a time direction and write $\mathbf{n}$ in Z^d as $(\mathbf{n}',n)$ where $\mathbf{n}'$ is in Z^{d-1} and n is in Z. We think of n as a time parameter. (Of course this is still just one of the space directions!) For each $h: Z^{d-1} \to C$ with $\Sigma_{\mathbf{n}'}|h(\mathbf{n}')|^2 = 1$ we define

$$\tilde{s}(n) = \sum_{n'} h(\mathbf{n}')s(\mathbf{n}',n)$$

Then $\tilde{s}$ is a random function of time.

The hypotheses of the theorem are the following. First, the spins must be highly aligned locally, in the sense that

$$E\left(|\tilde{s}(n+1) - \tilde{s}(n-1)|^2\right) \leqslant 2T$$

where T is small relative to $E(|s(\mathbf{n})|^2)$. Second, the random field must satisfy reflection positivity:

$$E\left[\left(\overline{\sum_{n\geqslant 0} c_n \tilde{s}(-n)}\right)\left(\sum_{n\geqslant 0} c_n \tilde{s}(n)\right)\right] \geqslant 0$$

Finally, we must have $d \geqslant 3$.

The conclusion of the theorem is that

$$E\big(s(\mathbf{0})s(\mathbf{n})\big) \to c > 0$$

as $\mathbf{n} \to \infty$ through even sites $\mathbf{n}$ in Z^d.

This statement of the theorem deserves some comment. First, the alignment estimate and the reflection positivity are valid for the ferromagnetic Ising model independently of the dimension d of space. Thus the role of the dimension of space is isolated in the last hypothesis.

Second, the alignment hypothesis is about separation by two time steps. This allows the axiomatic framework to encompass antiferromagnetic models, in which the spins tend to have alternating signs.

Third, the reflection positivity assumption looks mysterious, but its only use is to ensure a representation

$$E\left(\overline{\tilde{s}(0)}\tilde{s}(n)\right) = \langle f, R^{|n|}f\rangle$$

where $R = R^*$ is a self-adjoint operator (acting in some Hilbert space) with norm bounded by one, and where f is a vector in the Hilbert space. If in some application it is known that $\tilde{s}(n)$ is a component of a larger process that is Markov and time reversal invariant, then this representation may be read off directly. In any case the implication is not difficult to prove and may be found in a paper by Klein.[4]

Finally, the conclusion of the theorem refers to even sites, again only because of the possibility of antiferromagnetism.

It is worth mentioning why the conclusion implies the existence of multiple phases. There are two possibilities. If $E(s(\mathbf{n})) \neq 0$, then one applies symmetry to construct two measures where the two expectations have opposite sign. If $E(s(\mathbf{n})) = 0$, then the expression in the conclusion is $E(s(\mathbf{0})s(\mathbf{n})) = \mathrm{Cov}(s(\mathbf{0}), s(\mathbf{n}))$, the covariance in the sense of probability theory. The theorem says $\mathrm{Cov}(s(\mathbf{0}), s(\mathbf{n})) \to c > 0$ as $\mathbf{n} \to \infty$ through even sites. Thus there are long-range correlations. The sequence $s(\mathbf{n})$ of bounded functions has a subsequence that convergences weakly to some $s(\infty)$. It follows that $\mathrm{Cov}(s(\mathbf{0}), s(\infty)) = c > 0$. In particular $s(\infty)$ is not a constant. Thus we may define new probability measures by taking conditional probabilities given the sign of $s(\infty)$. These then give two different phases and the original phase is exhibited as a random choice of one of these two phases.

The proof uses Fourier analysis. The Fourier transform of the correlation function $E(s(\mathbf{0})s(\mathbf{n}))$ on Z^d is a measure on the torus T^d. This measure satisfies a bound that implies that it can have a singular part only at integer multiples of π. When $d \geqslant 3$ and the temperature T is sufficiently small, this component must be present and dominates the asymptotic behavior. A contribution from zero frequency produces long-range alignment, while a contribution from frequency π produces a staggered alignment. The reason the dimension of space comes in is that the infrared bound implies that the nonsingular part of the measure is integrable when $d > 2$.

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TWO-DIMENSIONAL ISING CORRELATIONS: CONVERGENCE OF THE SCALING LIMIT

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(What follows is the introduction to the authors' article in *Advances in Applied Mathematics* **2**:329–388 (1981).)

INTRODUCTION

In this paper we will establish formulas for the correlations of the two-dimensional Ising model in the absence of a magnetic field and prove the convergence of the scaling limit from above and below the critical temperature.

The theoretical developments which lead up to our results begin with Onsager's calculation of the free energy for this model in a classic 1944 paper.(52) Statistical mechanics in the finite-volume limit is expected to exhibit phase transitions through nonanalytic behavior in thermodynamic quantities; the Onsager formula for the free energy as a function of temperature was the first explicit example of such behavior. In a sequel to Onsager's paper, Kaufmann(34) simplified the analysis by emphasizing the role of the spin representations of the orthogonal group; Kaufman and Onsager(35) subsequently used this idea to study the short-range order. By 1949 Onsager(53) knew the formula for the spontaneous magnetization, and Yang gave an independent derivation of this result in 1952.(74)

In Ref. 28 Kac and Ward and later in Ref. 32 Kasteleyn pioneered a combinatorial attack on the Ising model. Montroll, Potts, and Ward(49) used this method to give formulas for the correlations as Pfaffians. The size of the Pfaffians in these formulas grows with the separation of the sites in the correlations and the asymptotic behavior at large separation (clustering) is far from evident. To go beyond the spontaneous magnetization in the analysis of the clustering of correlations, corrections to the Szegö formula were devised. This problem has a long history, and we mention in connection with the Ising model the fundamental papers by Wu(72) and by Kadanoff(29) in 1966, and by Cheng and Wu in 1967,(13) and refer to reader to the book by McCoy and Wu(40) for further details up to 1972.

In Fisher[18] and Kadanoff[30] a notion of scaling for statistical systems near a critical point was proposed. To understand the scaling limit for the Ising model it proved important to have formulas for the lattice correlations which manifested clustering explicitly. In 1973 the calculation of the two-point scaling function was announced in Refs 11 and 68 with details appearing in Ref. 73. Somewhat later several groups announced series expansion formulas for the scaled n-point functions.[4,10,43,57] McCoy *et al.*[43] employed Pfaffian techniques which evolved from the combinatorial approach to the Ising model (see also Ref. 41). The work of Sato *et al.*,[57] of Abraham,[2-5] and of Bariev[9,10] is more directly descended from the original algebraic approach of Onsager and Kaufman; an approach which, incidently, received further stimulus in the papers of Schultz *et al.*[65] and Kadanoff.[29]

In the passage to the scaling limit, the correlations become singular at points of coincidence. For example, the critical exponent specifying this singularity in the two-point function is "known" from the large-scale behavior at the critical temperature[40] (the two-point scaling function interpolates between the behavior at large separation at the critical temperature and the behavior at large separation away from the critical temperature). However, the precise asymptotics at short distance has never been directly computed from the known series expansions. This is not too surprising since these series were developed specifically to exhibit the behavior at large separation in the scaled distance. In Ref. 73 Wu, McCoy, Tracy, and Barouch found the precise short-distance asymptotics for the scaled two-point function by first showing that this function was expressible in terms of a Painlevé transcendent. Part of this analysis was put on a firmer footing in a later paper.[42]

The deeper reason for the occurrence of the Painlevé transcendent was first understood by Sato, Miwa, and Jimbo (SMJ).[58-63] They were aware that Painlevé transcendents occur naturally in the integration of Schlesinger's equations[64] for monodromy-preserving deformations of linear differential equations (oddly, the extensive work of Garnier[19] on this connection is not mentioned in the principal English reference, Ince[25]). In a remarkable series of papers, they developed new techniques in the theory of Clifford algebras,[59] generalized the monodromy idea to a partial differential equation (the Euclidean Dirac equation),[61] showed that the scaled n-point functions were the coefficients in the local expansion of a basis of multivalued solutions to the Euclidean Dirac equation,[62] and finally used this to demonstrate that the scaled n-point functions satisfy a nonlinear Pfaffian system of differential equations (every derivative is specified).[62,63] In the case of the two-point function, the Pfaffian system is integrable in terms of the particular Painlevé transcendent appearing in Refs. 42 and 73. A review of this work can be found in Ref. 26.

In the work we have described on the Ising model the level of mathematical rigor fluctuates considerably. In much of the work on the correlations, the subtleties of the boundary conditions for the infinite volume limit are side stepped. In all the work we are aware of there are "holes" of positive measure in the known regions of convergence for the series representations of the scaled n-point functions. In particular the important SMJ[62] analysis of the scaled correlations introduced (multivalued) continuum order–disorder correlations through complicated infinite series expansions whose known region of convergence has large gaps. The coefficients in the local expansions of these order–disorder correlations are identified as n-point functions again only at the level of the series expansions. One of the principal motivations for our paper is to lay the foundation for a treatment of the SMJ analysis in which the multivalued order–disorder correlations and the n-point functions appear as well-controlled limits of simply defined lattice analogs, and in which the local expansions are computed rigorously. We shall present this analysis in a forthcoming paper. Another important consideration for our work was to establish some of the expected probabilistic and field theoretic properties for the scaled n-point functions. Our contribution to these matters is presented in the final section of this paper.

In the first three sections of this paper, we prove (regularized) determinant formulas for the infinite-volume correlations (Theorems 2.1 and 3.2). The transfer matrix formalism in Section 1 permits us to express the correlations (with "plus" boundary conditions) for a semi-infinite box as the Fock expectation of a product in a finite-dimensional Clifford algebra. We apply results from Ref. 56 to give determinant formulas in this finite-dimensional situation and then prove the convergence of these determinants to the infinite-volume counterparts directly. Our proof is valid only below the critical temperature. Above the critical temperature we use a variant of Kramers–Wannier duality to relate the correlations with "open" boundary conditions to correlations of disorder variables (see Kadanoff and Ceva[31]) with "plus" boundary conditions below the critical temperature. This effectively reduces the convergence proof to the previous case and incidently identifies a natural disorder variable on the lattice. Once the determinant formulas are established, the infinite-dimensional results in Ref. 56 then give simple "abstract" characterizations of the infinite-volume correlations as Fock expectations (Theorem 2.2 and 3.3).

The use of "plus" boundary conditions permits us to use the convergence results[39] which show that the correlations obtained in the two-step infinite-volume limit natural for the transfer matrix approach are the same as the correlations which result from letting the sides of a square box tend simultaneously to infinity. This coincidence of limits establishes dihedral group invariance and that the correlations are the expectations of products of random fields,[14,15] neither of which properties is manifest in our explicit formulas.

In the fourth section we prove convergence of the scaling limit from above and below the critical temperature. Our formulas are not valid everywhere but the exceptional sets are measure zero. The resulting n-point scaling functions (below T_c) are given by formulas $\det_2(1 + G)$, where G is a Schmidt class operator.

In Section 5 we use Gaussian domination[51] and some integrability estimates for the two-point function to conclude that the correlations are locally integrable functions. We then use the Bochner–Minlos theorem to demonstrate that we have computed the n-point functions of a generalized random field.[16,20] The Osterwalder–Schrader axioms[54] are all direct consequences of the convergence of the scaling limit with the exception of rotational invariance. We do not prove rotational invariance; however, we note that McCoy and Wu have an unpublished demonstration of this property. Of particular interest in this last section are new formulas for the lattice two-point functions which we use to establish dominated convergence.

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For Baxter models see

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CRITICAL PHENOMENA AND THE RENORMALIZATION GROUP

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Basic phenomenology of phase transitions including definitions of critical exponents, scaling and definitions and results for *n-d* models were reviewed. The standard renormalization group 'recipe' was outlined. The exactly solvable one dimensional Ising and hierarchical models were briefly discussed. Expansions in ϵ where the lattice dimension is $4-\epsilon$ were mentioned but not discussed.

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